Black hole interacting with matter as a simple dynamical system

Petr Hájíček

Institute for Theoretical Physics
University of Bern
Sidlerstrasse 5, CH-3012 Bern
Switzerland

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Abstract

Recently, a variational principle has been derived from Einstein-Hilbert and a matter Lagrangian for the spherically symmetric system of a dust shell and a black hole. The so-called physical region of the phase space, which contains all physically meaningful states of the system defined by the variational principle, is specified; it has a complicated boundary. The principle is then transformed to new variables that remove some problems of the original formalism: the whole phase space is covered (in particular, the variables are regular at all horizons), the constraint has a polynomial form, and the constraint equation is uniquely solvable for two of the three conserved momenta. The solutions for the momenta are written down explicitly. The symmetry group of the system is studied. The equations of motion are derived from the transformed principle and are shown to be equivalent to the previous ones. Some lower-dimensional systems are constructed by exclusion of cyclic variables, and some of their properties are found.

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I. INTRODUCTION

The spherically symmetric gravitating shell has been used as a simplified quantum model at many occasions. One can mention the quantum effects of domain walls in the early Universe [1], quantum aspects of gravitational collapse [2], quantum theory of black holes [3] or Hawking evaporation of black holes [4].

The equations of motion for the shells has been derived by Dautcourt [5] and transformed to a geometric (i.e. "gauge invariant") form by Israel [6]. However, for a quantum theory, one needs a variational principle rather than dynamical equations. Such variational principle has as yet been just guessed from the dynamical equations, Refs. [7] and [8], or from some intermediate variational principle, Refs. [4] and [9]. Quite a number of different Hamiltonians have resulted and the corresponding quantum theories have not been unitarily equivalent. There are two sources of the ambiguity.

The first source of ambiguity is the invariance of the general relativity with respect to any coordinate transformations on one hand, and the property of Hamiltonians to generate dynamics with respect to a particular time coordinate on the other. For example, the choice of the proper time along the shell as such a coordinate leads to the $\cosh p$ Hamiltonian, Refs. [7] and [10], the Schwarzschild time coordinate inside the shell to the square-root Hamiltonian, Ref. [2], and the Schwarzschild time coordinate outside the shell to a merely implicitly determined Hamiltonian of Refs. [4] and [9]. An attempt to work out a gauge invariant theory for at least some class of the guessed Hamiltonians is Ref. [8]; based on a technical assumption that the super-Hamiltonian is quadratic in the momenta, it gives a unique action principle.

The second source of ambiguity is the fact that equations of motion do not determine the corresponding variational principle uniquely in general. This ambiguity can be removed by a direct derivation of the variational principle for the shell from the Einstein-Hilbert and shell matter Lagrangian. Three such derivations exist: Refs. [11], [12] and [13]; they all lead to the same variational principle for the same system. In Refs. [12] and [13], the reduction of the second order formalism to the spherically symmetric case is done first, followed by a reduction to dynamical variables of the shell alone and by the transformation to the second order formalism. In Ref. [11], a first and second order formalism for a general shell and gravitational field is derived (no symmetry); this second order formalism is reduced by the spherical symmetry and to shell variables in Ref. [14].

We are then left with the following dilemma. The super-Hamiltonian of Ref. [8] is simple and amenable to the existing gauge invariant quantization methods (like e.g., Refs. [17] and [18]). The super-Hamiltonian of Refs. [12], [14] and [13] follows from the Einstein-Hilbert and shell-matter Lagrangian, but it is extremely complicated: it contains nested square roots and hyperbolic functions, and for a good measure, it is formulated in coordinates that diverge on horizons. It would be difficult to quantize.

A natural question then arise: what is the relation between the guessed and simple variational principle of Ref. [8] and the derived but complicated one of Refs. [12], [14] and [13]? If the simple principle were equivalent to the complicated one, we could use the simple principle and forget the complicated one. However, the answer turns out to be rather surprising (see Ref. [15]): the two descriptions are only locally, but not globally equivalent. The local equivalence explains, why the equations of motion are the same. For the study

of the problem, a geometric approach of Ref. [16] has been applied. All gauge invariant properties of a constrained system are encoded in two objects: the constraint manifold Γ and the presymplectic form Ω on Γ. The equivalence of two such systems, (Γ_1, Ω_1) and (Γ_2, Ω_2) is then a well-defined mathematical concept independent of a choice of extended phase space, a choice of constraint functions or a choice of gauge.

In this situation, it seems natural to look for a transformation of the variational principle of Refs. [12], [14] and [13] to a better set of coordinates, and this will be the main topic of the present paper. The method will again be based on the gauge invariant description (Γ, Ω) of reparametrization invariant systems. We shall perform the transformation in three steps; to motivate the steps, we take into account the symmetry of the system, the structure of the so-called *Cartan form* Θ, which is defined by $d\Theta := \Omega$, and the topology of the space (Γ, Ω). The topological problem involved here is the existence of transversal surfaces in Γ; such surfaces are nowhere tangential to the dynamical trajectories. Functions whose levels are transversal are not only helpful, if one looks for nice coordinates, because they simplify Θ. Quite generally, they have to do with the existence of Hamiltonian and with the possibility to give the quantum dynamics the form of Schrödinger equation (cf. Ref. [16]). We shall, therefore, also look for transversal surfaces in a more systematic way. Finally, we shall find that the structure of the phase space of the spherically symmetric shell is not simple. Only a proper subset, which we call *physical region*, contains physical states of the shell. Its nature can be roughly specified as follows. All points inside it correspond to the system consisting of a shell of positive mass and radius, interacting with a black hole. The boundary of the region is complicated; it contains shells with zero radius, shells with zero rest mass (which cannot, however, be regarded as shells made of light-like matter), as well as self-gravitating isolated shells with positive radius and mass—these are the points at the boundary that are physically meaningful. The points outside the region that describe shells with positive radius and mass correspond to systems consisting of the shell and a negative mass source. A conclusion seems to be that the system will be difficult to quantize even if the new variables has made it algebraically simple. One problem will be to keep the spectra of observables contained within the physical region. Another problem will be to construct a unitary dynamics, because the classical dynamics breaks down at the boundary of the physical region.

The plan of the paper is as follows. In Sec. II, we collect the relevant results of the previous papers, mainly Refs. [14] and [13]. The variational principle that we describe allows also for an internal degree of freedom of the shell; it depends, therefore, of two additional variables in comparison with Refs. [12] and [14]: the proper time along the shell trajectory and the rest mass of the shell. We determine the constraint manifold Γ and the Cartan form Θ that follow from the variational principle. We study the physical region.

In Sec. III, we introduce three new functions on the phase space that are regular at the horizons and that will replace the momentum P of the shell and the two Schwarzschild times $T^{\pm}$ that diverge there. We describe their physical and geometrical meaning, using also some results of previous papers. In Sec. IV, we specify a new set of coordinates on Γ, and show that the transformation from the old to the new coordinates is differentiable and invertible. We also find the ranges of these coordinates in the physical region. Sec. V collects and extends the results about symmetries of the system that were obtained in Ref. [15] and expresses the symmetry transformations in the new coordinates; the form of the

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transformation simplifies. The new times are cyclic coordinates as the old have been. In Sec. VI, we transform the Cartan form into new coordinates. The square roots and hyperbolic functions disappear, and an equivalent form becomes regular at the horizons so that it can be smoothly matched accross them. However, the Cartan form is still complicated and this motivates further transformations. The final result thereof can be described as follows. The extended phase space is $(\mathbf{R}^8, \bar{\Omega})$. The natural coordinates of $\mathbf{R}^8$ form a Darboux chart for $\bar{\Omega}$. The equation of the constraint surface Γ in $\mathbf{R}^8$ is polynomial in all momenta and the presymplectic form $d\Theta$ is the pull-back of $\bar{\Omega}$ to Γ.

In Sec. VII, we study the polynomial constraint. We show that it defines two disjoint submanifolds in $\mathbf{R}^8$. One, denoted by Γ_6, is six-dimensional and lies well outside the physical region. The other, denoted by Γ_7, is seven-dimensional and intersects the physical region. In the intersection of Γ_7 with the physical region, we find two different foliations by transversal surfaces. They are not globally transversal, however, because none of them is intersected by all dynamical trajectories. The problem is that each dynamical trajectory starts or finishes at the $(R = 0)$-subset of the boundary of the physical region. The two different Hamiltonians coresponding to the two foliation are written down explicitly; they contain second and third roots.

In Sec. VIII, we repeat the study for some important lower-dimensional cases; they can be obtained by exclusion of cyclic coordinates from the original system. In particular, a (dynamical) black hole and the shell without its internal degree of freedom, the shell in the field of a fixed black hole external field, or the isolated self-gravitating shell (with the flat spacetime inside) are considered.

II. DESCRIPTION OF THE SYSTEM

In this section, we shall collect some results of Refs. [8], [14] and [13] so that the paper becomes relatively self-consistent. We shall also describe and study the physical region.

A spherically symmetric thin-shell spacetime solution of Einstein equations can be constructed as follows. Consider two Schwarzschild spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$ with Schwarzschild masses E_1 and E_2. Let Σ_1 be a timelike hypersurface in $\mathcal{M}_1$ and Σ_2 be one in $\mathcal{M}_2$. Let Σ_1 divide $\mathcal{M}_1$ into two subspacetimes, $\mathcal{M}_{1+}$ and $\mathcal{M}_{1-}$, and similarly Σ_2 divide $\mathcal{M}_2$ into $\mathcal{M}_{2+}$ and $\mathcal{M}_{2-}$. As everything is spherically symmetric, all spacetimes are effectively two-dimensional; we chose fixed time and space orientation in the two-dimensional Schwarzschild spacetimes so that future and past as well as right and left are unambiguous; let then $\mathcal{M}_{2+}$ and $\mathcal{M}_{1+}$ be right with respect to $\mathcal{M}_{2-}$ and $\mathcal{M}_{1-}$. Let Σ_1 and Σ_2 be isometric; then the spacetime $\mathcal{M}_{1-}$ can be pasted together with the spacetime $\mathcal{M}_{2+}$ along the boundaries Σ_1 and Σ_2. The result is a shell spacetime $\mathcal{M}_s$. Given a shell spacetime, we shall leave out the indices 1 and 2, having right (left) energy E_+ (E_-), shell trajectory Σ, and the right (left) subspacetime $\mathcal{M}_+$ ($\mathcal{M}_-$). Thus, $\mathcal{M}_+ = \mathcal{M}_2 \cap \mathcal{M}_s$ and $\mathcal{M}_- = \mathcal{M}_1 \cap \mathcal{M}_s$. In Refs. [14], [15] and [8], the Schwarzschild masses E_ϵ, $\epsilon = \pm 1$ were assumed to be *positive*.

Each subspacetime $\mathcal{M}_\epsilon$ has the metric

$$ds_\epsilon^2 = -F_\epsilon(R)(dT^\epsilon)^2 + F_\epsilon^{-1}(R)dR^2, \tag{1}$$

where

$$F_\epsilon(R) := 1 - \frac{2E_\epsilon}{R},$$

and it is split up by its horizons into four quadrants: we denote by Q_I that which is adjacent to the right infinity, Q_{II} to the left infinity, Q_{III} to the future singularity and Q_{IV} to the past singularity. Suitable notation (introduced in Ref. [14]) enables us to write all formulas in a form valid in any quadrant: we define four sign functions a_ϵ and b_ϵ distinguishing the quandrants of $\mathcal{M}_\epsilon$: $a_\epsilon := \mathrm{sgn} F_\epsilon$, and b_ϵ equals to $+1$ (-1) in the past (future) of the event horizon in the spacetime $(\mathcal{M}_\epsilon, g_\epsilon)$. We also include the dust internal degree of freedom, M, and the proper time T along the shell trajectory using the results of Ref. [13]. Then the total action for the dust and gravity can be rewritten in the form

$$S_\Sigma[\mathsf{T}, \mathsf{M}; T^\pm, E_\pm; R, P; \nu] = \int dt (P\dot{R} - E_+\dot{T}^+ + E_-\dot{T}^- + \mathsf{M}\dot{\mathsf{T}} - \nu\mathcal{C}), \qquad (2)$$

where ν is a Lagrange multiplier,

$$\mathcal{C} = \mathsf{M} - R\sqrt{F_+ + F_- - 2a_-b_+b_-\sqrt{|F_+F_-|}\,\mathrm{sh}_{a_+a_-}\frac{P}{R}}, \qquad (3)$$

is the super-Hamiltonian, $\mathrm{sh}_a x := (\mathrm{e}^x + a\mathrm{e}^{-x})/2$ for any $a = \pm 1$ and $x \in (-\infty, \infty)$. The momentum P conjugate to the radial coordinate R can be defined as follows (for more detail, see Ref. [14]). Let (n_ϵ, m_ϵ) be the orthonormal dyad at a shell point in the subspacetime $(\mathcal{M}_\epsilon, g_\epsilon)$, the vector n_ϵ being tangential to the shell and future oriented, m_ϵ being right oriented; we call it *shell dyad*. Let further the so-called *Schwarzschild dyad* $(n_{S\epsilon}, m_{S\epsilon})$ be defined at each point of the shell in the spacetime $(\mathcal{M}_\epsilon, g_\epsilon)$ by the requirement that $n_{S\epsilon}$ be timelike future oriented, $m_{S\epsilon}$ be right oriented and that one of the vectors be tangential to the $(R = \mathrm{const})$-curves. Clearly, in each quadrant, we have a different formula for the components of the Schwarzschild dyad; with respect to the Schwarzschild coordinates,

$$\text{for } F_\epsilon > 0, \quad n_{S\epsilon} = (b_\epsilon/\sqrt{F_\epsilon}, 0), \quad m_{S\epsilon} = (0, b_\epsilon\sqrt{F_\epsilon});$$
$$\text{for } F_\epsilon < 0, \quad n_{S\epsilon} = (0, b_\epsilon\sqrt{|F_\epsilon|}), \quad m_{S\epsilon} = (-b_\epsilon/\sqrt{|F_\epsilon|}, 0).$$

Then, P_ϵ/R is the hyperbolic angle between the Schwarzschild and shell dyads:

$$n_\epsilon = n_{S\epsilon}\cosh P_\epsilon/R + m_{S\epsilon}\sinh P_\epsilon/R,$$
$$m_\epsilon = n_{S\epsilon}\sinh P_\epsilon/R + m_{S\epsilon}\cosh P_\epsilon/R.$$

Finally, $P := [P]$, where we use the common short-hand for the jump $[A] := A_+ - A_-$ of a quantity A across the shell.

The extended phase space of the system is eight-dimensional, split up into 16 disjoint sectors, each sector being a pair of quadrants, one chosen from the left and one from the right subspacetimes. Each of these sectors can be covered with the coordinates P, R, E_+, T^+, E_-, T^-, M and T. The constraint surface that we call Γ is defined by the constraint equation $\mathcal{C} = 0$ in the extended phase space. It does not intersects all 16 sectors, but it is

split up into at most (if $E_- > \mathsf{M}$) non-empty intersections of the sectors with it.[1] These can be covered by the system of seven coordinates P, R, E_+, T^+, E_-, T^- and T. The following coordinates are assumed to have non-trivial ranges:

$$R > 0, \quad E_+ > 0, \quad E_- > 0, \quad \mathsf{M} > 0. \tag{4}$$

The pull back of the Liouville form of the action (2) to the constraint surface Γ in these coordinates is simply

$$\Theta = PdR - E_+ dT^+ + E_- dT^- + \mathsf{M}d\mathsf{T}, \tag{5}$$

where M is now a function of the seven coordinates defined by

$$\mathsf{M} = R\sqrt{F_+ + F_- - 2a_-b_+b_-\sqrt{|F_+F_-|}\,\mathrm{sh}_{a_+a_-}\frac{P}{R}}. \tag{6}$$

The form (5) is called *Cartan form* of the system; it contains all information about the dynamics and about the Poisson brackets (see, e.g. Refs. [19] or [15]).

The present paper starts from these formulas and quantities. The first new observation that we make is, that all these formulae remain valid also for the values $E_+ = 0$ or $E_- = 0$, if the notation and conventions are adapted a little. These cases are interesting because they describe an isolated self-gravitating shell (with the flat spacetime inside). Indeed, the metric of the flat spacetime is covered by formula (1). However, the flat spacetime does not contain any horizon and is not split into quadrants. Still, a piece of the flat spacetime can be used in four distinct ways in the contruction of the shell spacetime: either it lies to the left ($E_- = 0$) or to the right ($E_+ = 0$) of the shell, and, in each case, the R coordinate can either increase to the right or to the left (R is always spacelike in the flat spacetime). We can, therefore, formally define the *E-spacetime* for $E > 0$ as before to be the Kruskal spacetime of Schwarzschild mass E with four quadrants, and for $E = 0$ as consisting of two topologically separated quadrants Q_I and Q_{II}, each isometric to the flat spacetime, but with oposite time and space orientations: in Q_I, the radial ("Schwarzschild") coordinate R increases to the right and the "Schwarzschild" time T to the future; in Q_{II}, R increases to the left and T to the past. It is amusing to observe that the validity of the formulae (2) and (3) can then be extended to vanishing E's, if we just use the values $a_\epsilon = 1$ everywhere, $b_\epsilon = +1$ in Q_I and $b_\epsilon = -1$ in Q_{II}. This works because the components of the Schwarzschild dyad that are crucial for the derivation of Eq. (3) in Ref. [14] retain their form in the quadrants Q_I and Q_{II} of the E-spacetime for $E = 0$, and there are no quadrants Q_{III} and Q_{IV}. Similarly, the sign relation (14) of Ref. [8], which is basic for the form of the dynamical equations, remains valid with our orientation and quadrant conventions. Observe that the presence of two topologically separated subspacetimes is not observable and does not imply any physical

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[1]Observe that we admit unphysical sectors, i.e. those, in which the constraint equation $\mathcal{C} = 0$ has no solutions. This simplifies the boundary of the extended phase space a great deal and does not lead to any problem: it is the very purpose of extended phase spaces to enclose also non-physical poits, if this leads to any simplification.

assertion; similarly, the orientation of coordinates is a mere coordinate convention without any physical meaning.

A very important but rather embarrasing observation is now that a formally impeccable shell spacetime can be pasted together from Schwarzschild spacetimes of *any* mass, even a negative one. The metric is still given by Eq. (1) and the spacetime has a similar global structure as the flat one: there are no horizons and there are two possible orientations of how it can be built in. Again, all equations remain valid, if we accept the same sign, quadrant and orientation conventions as for the flat spacetime and define the (two-quadrant) E-spacetime for $E < 0$ analogously to what we have done for $E = 0$. The corresponding shell dynamics is then again given by the action (2).

Let us quickly discuss the dynamical equations following from the action (2) under these conventions. There are three conservation laws,

$$\dot{E}_+ = \dot{E}_- = \dot{\mathsf{M}} = 0, \tag{7}$$

resulting from varying the action with respect to the cyclic coordinates T^+, T^- and T. The variations with respect of the conserved momenta and P with the subsequent simplification by the constraint (6) yield:

$$\dot{\mathsf{T}} = \nu, \tag{8}$$

$$\dot{T^\epsilon} = -\epsilon\nu\frac{R}{\mathsf{M}}\left(1 - a_-b_+b_-\frac{\sqrt{|F_+F_-|}}{F_\epsilon}\,\mathrm{sh}_{a_+a_-}\frac{P}{R}\right), \tag{9}$$

$$\dot{R} = \nu a_-b_+b_-\frac{R}{\mathsf{M}}\sqrt{|F_+F_-|}\,\mathrm{sh}_{-a_+a_-}\frac{P}{R}. \tag{10}$$

The equation determining $\dot{P}$ can be obtained either by varying the action with respect to R, or by differentiating the constraint equation $\mathcal{C} = 0$ with respect to t followed by substitution for all other t-derivatives from Eqs. (7)–(10). Let us rewrite the constraint equation in the form

$$-\frac{\mathsf{M}}{2R} + \frac{R}{2\mathsf{M}}(F_+ + F_-) = a_-b_+b_-\frac{R}{\mathsf{M}}\sqrt{|F_+F_-|}\,\mathrm{sh}_{a_+a_-}\frac{P}{R}. \tag{11}$$

Eqs. (7)–(11) represent a complete system of dynamical equations for the shell.

We derive some important consequences from the dynamical equations. Eqs. (8)–(11) imply that

$$-F_\epsilon\left(\frac{dT^\epsilon}{d\mathsf{T}}\right)^2 + \frac{1}{F_\epsilon}\left(\frac{dR}{d\mathsf{T}}\right)^2 = -1,$$

confirming that T is the proper time along shell trajectories. Eqs. (8), (9) and (11) deliver the *time equation*:

$$F_\epsilon\frac{dT^\epsilon}{d\mathsf{T}} = -\epsilon\frac{\mathsf{M}}{2R} + \frac{E_+ - E_-}{\mathsf{M}}. \tag{12}$$

From Eqs. (10) and (11), the *radial equation* follows,

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$$\frac{dR}{d\mathsf{T}} = -\omega\sqrt{-V}, \tag{13}$$

where

$$V(R) = -\frac{\mathsf{M}^2}{4R^2} - \frac{E_+ + E_-}{R} - \frac{(E_+ - E_-)^2}{\mathsf{M}^2} + 1 \tag{14}$$

and $\omega = \pm 1$ is a suitable sign (see Eq. (2) of Ref. [15]). The square of the radial equation can be rewritten in one of the two forms

$$F_\epsilon + \left(\frac{dR}{d\mathsf{T}}\right)^2 = J_\epsilon^2, \tag{15}$$

according as $\epsilon = \pm 1$, where

$$J_\epsilon := \frac{\mathsf{M}}{2R} - \epsilon\frac{E_+ - E_-}{\mathsf{M}} \tag{16}$$

(cf. Ref. [8]). It follows immediately from this definition of J_ϵ that

$$J_+ + J_- = \frac{\mathsf{M}}{R},$$

and this can be transformed with the help of Eq. (15) to

$$\mathrm{sgn}J_+\sqrt{F_+ + \left(\frac{dR}{d\mathsf{T}}\right)^2} + \mathrm{sgn}J_-\sqrt{F_- + \left(\frac{dR}{d\mathsf{T}}\right)^2} = \frac{\mathsf{M}}{R}.$$

However, comparing Eqs. (12) and (16), we observe that

$$\mathrm{sgn}J_\epsilon = -\epsilon\,\mathrm{sgn}\left(F_\epsilon\frac{dT^\epsilon}{d\mathsf{T}}\right).$$

The sign of the expression $F_\epsilon(dT^\epsilon/d\mathsf{T})$ deserves a name; let us call it τ_ϵ (cf. [8]). Then, we obtain finally:

$$-\tau_+\sqrt{F_+ + \left(\frac{dR}{d\mathsf{T}}\right)^2} + \tau_-\sqrt{F_- + \left(\frac{dR}{d\mathsf{T}}\right)^2} = \frac{\mathsf{M}}{R}. \tag{17}$$

This is the so-called *Israel equation* for dust shell (cf. Eqs. (14), (21) and (22) of Ref. [8]).

Eqs. (12)–(17) are valid for all values of Schwarzschild masses $E_+ \in (-\infty, +\infty)$ and $E_- \in (-\infty, +\infty)$. Some features of the unphysical (negative E_ϵ) cases are described by the following theorem.

Theorem 1 *Let E_ϵ be non-positive and not larger than $E_{-\epsilon}$ for some value of ϵ. Then the timelike (connected) center ($R = 0$) curve lies in the subspacetime $\mathcal{M}_\epsilon$ on the ϵ-side of the shell.*

To prove the theorem, we observe that $E_\epsilon \leq E_{-\epsilon}$ implies

$$\sqrt{F_\epsilon + \left(\frac{dR}{d\mathsf{T}}\right)^2} \geq \sqrt{F_{-\epsilon} + \left(\frac{dR}{d\mathsf{T}}\right)^2}.$$

Thus, the left-hand side of Eq. (17) can only be positive, if $\tau_\epsilon = -\epsilon$. The assumption $E_\epsilon \leq 0$ of the theorem implies that $\mathcal{M}_\epsilon$ is the two-quadrant E-spacetime with $\tau_\epsilon > 0$ in the quadrant Q_I and $\tau_\epsilon < 0$ in the quadrant Q_{II}; this follows from the definition of τ_ϵ and from the positivity of F_ϵ (in fact, for $E_\epsilon \leq 0$, we have $F_\epsilon \geq 1$). Hence, to the right ($\mathcal{M}_+$) of the shell, there is a part of Q_{II} with the center to the right, and to the left of the shell ($\mathcal{M}_-$), there must be Q_I with the center to the left, Q. E. D.

An interpretation of the theorem is that the shell spacetime containing one or two negative-mass subspacetimes is unphysical: there must be at least one negative mass source somewhere outside the shell. The theorem also says that a shell spacetime constructed from one flat and one positive-mass subspacetimes must have a regular center inside the flat subspacetime.

To summarize: The action (2) generates a regular dynamics in the enlarged phase space defined just by

$$\mathsf{M} > 0, \quad R > 0. \tag{18}$$

For $E_+ > 0$ and $E_- > 0$, it describes a shell interacting with a black hole of mass E_- (or E_+, depending on from where we observe the shell). For $E_+ > 0$ and $E_- = 0$ (or $E_+ = 0$ and $E_- > 0$), it describes a self-gravitating shell with flat spacetime inside. If any of E_+ and E_- are negative, it describes the shell interacting with a negative mass source. Only the points with

$$E_+ \geq 0, \quad E_- \geq 0 \tag{19}$$

are physically sensible; the subset specified by the inequalities (18) and (19) is the *physical region* of the phase space. There seems to be nothing about the variational principle (2) that would help us to enforce the validity of the Eqs. (19) in any natural, automatic way.

III. RADIAL AND KRUSKAL MOMENTA

From the definition of the functions $T^\pm$ and P as given in the previous section, it follows that they are singular at the horizons: T^ϵ at $R = 2E_\epsilon$ and P at $R = 2E_-$ as well as at $R = 2E_+$. In this section, we introduce three functions that are well-defined on the whole constraint surface to replace $T^\pm$ and P. Another replacement of $T^\pm$ and P with similar properties has been tried in Ref. [14]. There, the functions were constructed from the Kruskal coordinates; however, the Kruskal coordinates make no sense for flat spacetime and so the important case $E_- = 0$ could not be incorporated. In this subsection, we remove this problem.

Let us define the new functions q, $[T_1]$ and $\bar{T}_1$ by

$$q := a_- b_+ b_- R \sqrt{|F_+ F_-|} \operatorname{sh}_{-a_+ a_-} \frac{P}{R}, \tag{20}$$

$$\bar{T}_1 := \frac{T^+ + T^-}{2} + E_+ \ln \left| \frac{u_+}{v_+} \right| + E_- \ln \left| \frac{u_-}{v_-} \right|, \tag{21}$$

$$[T_1] := T^+ - T^- + 2E_+ \ln \left| \frac{u_+}{v_+} \right| - 2E_- \ln \left| \frac{u_-}{v_-} \right|, \tag{22}$$

and the auxiliary functions u_ϵ and v_ϵ by

$$u_\epsilon := -\frac{q}{\mathsf{M}} - \frac{[E]}{\mathsf{M}} + \epsilon \frac{\mathsf{M}}{2R}, \quad v_\epsilon := -\frac{q}{\mathsf{M}} + \frac{[E]}{\mathsf{M}} - \epsilon \frac{\mathsf{M}}{2R}, \tag{23}$$

in Eq. (23), the functions q and M are given by Eqs. (20), (6) and $[E] := E_+ - E_-$.

The meaning of the momentum q can be seen, if it is expressed by means of velocities along dynamical trajectories. Comparing Eqs. (8), (10) and (20), we find that

$$q = \mathsf{M} \frac{dR}{d\mathsf{T}}; \tag{24}$$

hence, q is a kind of radial momentum, and it is regular everywhere along dynamical trajectories.

The meaning of the functions $\bar{T}_1$ and $[T_1]$ can be inferred from that of T_1^ϵ, which are defined by

$$T_1^\epsilon := \bar{T}_1 + \frac{\epsilon}{2}[T_1].$$

Eqs. (21) and (22) lead then to

$$T_1^\epsilon = T^\epsilon + 2E_\epsilon \ln \left| \frac{u_\epsilon}{v_\epsilon} \right|$$

for each $\epsilon = \pm 1$. Let us limit ourselves to the physical region and distinguish two cases. $E_\epsilon = 0$. Then, $T_1^\epsilon = T^\epsilon$ in both quadrants of the E-spacetime, so $b_\epsilon T_1^\epsilon$ is the time coordinate of the inertial system of the Minkowski spacetime $\mathcal{M}_\epsilon$ in which the center of mass of the shell is in rest. $E_\epsilon > 0$. Then, it is possible to introduce Kruskal coordinates U_ϵ and V_ϵ in the Kruskal spacetime $\mathcal{M}_\epsilon$ by

$$R = 2E_\epsilon \kappa(-U_\epsilon V_\epsilon),$$
$$T^\epsilon = 2E_\epsilon \ln \left| \frac{V_\epsilon}{U_\epsilon} \right| \tag{25}$$

and by the requirement that both functions U_ϵ and V_ϵ increse to the future, where κ is the well-known monotonous function defined on the interval $(-1, \infty)$ by its inverse,

$$\kappa^{-1}(x) = (x - 1)\mathrm{e}^x.$$

At each point of the shell trajectory $U_\epsilon = U_\epsilon(t)$ and $V_\epsilon = V_\epsilon(t)$, the so-called *Kruskal momentum*, P_K^ϵ, was defined in Ref. [14] by

$$P_K^\epsilon := \frac{R}{2} \ln \frac{dV_\epsilon}{dU_\epsilon}.$$

In Ref. [15], four further functions u_ϵ and v_ϵ were defined by

$$u_\epsilon := \frac{U_\epsilon e^{P_K^\epsilon/R}}{\sqrt{\kappa_\epsilon e^{\kappa_\epsilon}}}, \quad v_\epsilon := \frac{V_\epsilon e^{-P_K^\epsilon/R}}{\sqrt{\kappa_\epsilon e^{\kappa_\epsilon}}}, \tag{26}$$

where $\kappa_\epsilon := \kappa(-U_\epsilon V_\epsilon)$.

The following relation between the pairs u_ϵ, v_ϵ and R, E_ϵ was shown to hold in Ref. [15] (Eqs. (32) and (33)):

$$u_\epsilon = -\frac{[E]}{\mathsf{M}} + \epsilon \frac{\mathsf{M}}{2R} + \omega\sqrt{-V},$$
$$v_\epsilon = \frac{[E]}{\mathsf{M}} - \epsilon \frac{\mathsf{M}}{2R} + \omega\sqrt{-V}.$$

A comparison with Eqs. (13) and (14) shows that our functions u_ϵ and v_ϵ defined by Eqs. (23) coincide with the functions u_ϵ and v_ϵ of Ref. [15], if the dynamical equations are satisfied. Eq. (26) then shows that u_ϵ and v_ϵ vanish at horizons and that their logarithms diverge there. It then also follows for the functions T_1^ϵ along dynamical trajectories that

$$T_1^\epsilon = \frac{4E_\epsilon P_K^\epsilon}{R}. \tag{27}$$

We can see that the functions T_1^ϵ are directly determined by the geometry of the dynamical trajectory in the spacetime $\mathcal{M}_\epsilon$ and are regular everywhere along the trajectory.

IV. A REGULAR COORDINATE SYSTEM

In this section, we describe a coordinate system that is regular at the horizons, and at the same time remains meaningful for the special values $E_\epsilon = 0$ of Schwarzschild masses.

As such coordinates on Γ, we suggest the seven functions $q, R, \mathsf{M}, \mathsf{T}, [E], \bar{T}_1$, and $[T_1]$. Let us show that the Jacobian of the transformation from these coordinates to P, R, E_+, T^+, E_-, T^- and T is non-zero. To calculate the determinant, we observe that the four columns R, T^+, T^- and T of the Jacobian

$$\frac{\partial(q, \dots, [T_1])}{\partial(P, \dots, \mathsf{T})}$$

contain each at most two non-zero elements; if we expand the determinant along these columns, we obtain

$$\frac{\partial(q, \dots, [T_1])}{\partial(P, \dots, \mathsf{T})} = -\frac{\partial q}{\partial P}\left(\frac{\partial \mathsf{M}}{\partial E_+} + \frac{\partial \mathsf{M}}{\partial E_-}\right) + \frac{\partial \mathsf{M}}{\partial P}\left(\frac{\partial q}{\partial E_+} + \frac{\partial q}{\partial E_-}\right). \tag{28}$$

For the derivatives involved here, Eqs. (6) and (20) yield immediately:

$$\frac{\partial q}{\partial P} = a_-b_+b_-\sqrt{|F_+F_-|}\mathrm{sh}_{a_+a_-}\frac{P}{R}, \quad \frac{\partial \mathsf{M}}{\partial P} = -a_-b_+b_-\sqrt{|F_+F_-|}\frac{R}{\mathsf{M}}\mathrm{sh}_{-a_+a_-}\frac{P}{R},$$

$$\frac{\partial \mathsf{M}}{\partial E_+} + \frac{\partial \mathsf{M}}{\partial E_-} = \frac{R}{\mathsf{M}}\left(-2 + a_+b_+b_-\frac{F_+ + F_-}{\sqrt{|F_+F_-|}}\mathrm{sh}_{a_+a_-}\frac{P}{R}\right),$$

$$\frac{\partial q}{\partial E_+} + \frac{\partial q}{\partial E_-} = -a_+b_+b_-\frac{F_+ + F_-}{\sqrt{|F_+F_-|}}\mathrm{sh}_{-a_+a_-}\frac{P}{R}.$$

Substituting this in Eq. (28) and using again Eq. (6), we arrive at the simple result

$$\frac{\partial(q, \ldots, [T_1])}{\partial(P, \ldots, \mathsf{T})} = -\frac{\mathsf{M}}{R},$$

which holds in each sector of Γ.

It follows that the transformation is regular within each sector and can be inverted. In fact, it is not difficult to find the functions defining the inverse transformation. First, we show that

$$\bar{E} = \frac{R}{2}\left(1 + \frac{q^2}{\mathsf{M}^2} - \frac{[E]^2}{\mathsf{M}^2} - \frac{\mathsf{M}^2}{4R^2}\right) \tag{29}$$

in each sector. To this aim, we substitute for q and M from Eqs. (6) and (20) into Eq. (29) and use the identities

$$F_+ + F_- = 2\left(1 - \frac{2\bar{E}}{R}\right), \quad F_+F_- = \left(1 - \frac{2\bar{E}}{R}\right)^2 - \frac{[E]^2}{R^2}. \tag{30}$$

In this way, we obtain

$$\frac{2\bar{E}}{R} - 1 = \frac{R^2}{\mathsf{M}^2}\left(-2(1 - 2\bar{E}/R)^2 + 2a_-b_+b_-(1 - 2\bar{E}/R)\sqrt{|F_+F_-|}\mathrm{sh}_{a_+a_-}\frac{P}{R}\right).$$

Now, the equality follows immediately from Eq. (6).

The next non-trivial part of the inverse transformation is the relation

$$P = \frac{R}{2}\ln\left|\frac{u_+v_-}{u_-v_+}\right|, \tag{31}$$

which again holds in each sector. To show Eq. (31), we use Eqs. (23) to obtain

$$\frac{u_+v_-}{u_-v_+} = \frac{\left(q - \frac{\mathsf{M}^2}{2R}\right)^2 - [E]^2}{\left(q + \frac{\mathsf{M}^2}{2R}\right)^2 - [E]^2}.$$

The substitution for q and M gives

$$q + \eta\frac{\mathsf{M}^2}{2R} = \eta(R - 2\bar{E}) - \eta a_{-\eta}b_+b_-R\sqrt{|F_+F_-|}e^{\eta P/R},$$

where $\eta = \pm 1$. After some rearrangement and applying Eqs. (30), we have

$$\left(q + \eta\frac{\mathsf{M}^2}{2R}\right)^2 - [E]^2 = -a_{-\eta}b_+b_-R^2\sqrt{|F_+F_-|}\,\mathrm{e}^{\eta P/R}\left(F_+ + F_- - 2a_-b_+b_+\mathrm{sh}_{a_+a_-}\frac{P}{R}\right).$$

Then, Eq. (6) leads immediately to

$$\left(q + \eta\frac{\mathsf{M}^2}{2R}\right)^2 - [E]^2 = -a_{\eta}b_+b_-\mathsf{M}^2\sqrt{|F_+F_-|}\,\mathrm{e}^{\eta P/R},$$

and this implies Eq. (31).

The rest of the inverse transformation is easy: the functions R and T are the same in both sets of variables,

$$E_\epsilon = \bar{E} + \frac{\epsilon}{2}[E] \tag{32}$$

and

$$T^\epsilon = \bar{T}_1 + \frac{\epsilon}{2}[T_1] - 2E_\epsilon \ln\left|\frac{u_\epsilon}{v_\epsilon}\right|; \tag{33}$$

in the last equation, one has to substitute Eq. (32) for E_ϵ and Eqs. (23) for u_ϵ and v_ϵ.

The ranges of the variables q, R, M, T, $[E]$, $\bar{T}_1$, and $[T_1]$ in the physical region are implied by the conditions (18) (19) and Eq. (29). Let us work them out. Eq. (29) together with $\bar{E} \geq 0$ implies that

$$[E]^2 \leq \mathsf{M}^2 + q^2 - \frac{\mathsf{M}^4}{4R^2}; \tag{34}$$

hence,

$$\mathsf{M}^2 + q^2 \geq \frac{\mathsf{M}^4}{4R^2}. \tag{35}$$

The two inequalities $E_+ \geq 0$ and $E_- \geq 0$ are equivalent to $[E]^2 \leq 4\bar{E}^2$. Eq. (29) implies that this inequality is, in turn, equivalent to

$$\left([E]^2 - \left(\mathsf{M}^2 + q^2 + \frac{\mathsf{M}^4}{4R^2}\right)\right)^2 \geq \left(\frac{\mathsf{M}^2}{R}\sqrt{\mathsf{M}^2 + q^2}\right)^2. \tag{36}$$

From Eq. (34), it follows that

$$[E]^2 - \left(\mathsf{M}^2 + q^2 + \frac{\mathsf{M}^4}{4R^2}\right) \leq -\frac{\mathsf{M}^4}{2R^2} < 0.$$

Eq. (36) is, therefore, equivalent to

$$[E]^2 \leq \left(\sqrt{\mathsf{M}^2 + q^2} - \frac{\mathsf{M}^2}{2R}\right)^2. \tag{37}$$

Eq. (35) shows that the inequality (37) is not weaker than (34). Hence, all information is contained in the following inequality:

$$|[E]| \leq \sqrt{\mathsf{M}^2 + q^2} - \frac{\mathsf{M}^2}{2R}. \tag{38}$$

Inequality (38) together with $\mathsf{M} > 0$ and $R > 0$ define the ranges of the variables q, R, M, T, $[E]$, $\bar{T}_1$, and $[T_1]$ in the physical region.

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V. SYMMETRY OF THE SYSTEM

In this section, we collect the results of Ref. [15] on the symmetry of the shell action and describe the action of the symmetry transformations on the new variables.

In Ref. [15], we have found that there is a continuous symmetry group generated by an arbitrary function of E_+ and E_-. A simple generalization of the argument given in Ref. [15] implies that the following finite transformation of the variables E_ϵ, u_ϵ, v_ϵ and $\tilde{P}_\epsilon$, which were used there, is a symmetry of the system:

$$\tilde{P}_\epsilon \mapsto \tilde{P}_\epsilon - \epsilon \frac{\partial \Lambda}{4 \partial E_\epsilon}, \quad \epsilon = \pm 1,$$

the other variables being invariant, where $\Lambda(E_+, E_-)$ is an arbitrary smooth function of two variables (the factor $1/4$ is introduced for convenience). In addition to this continuous infinitely dimensional group, there were two reflections, which we denote here by σ_1 and σ_2. σ_1 is a time reflection defined by

$$E_\epsilon \mapsto E_\epsilon, \quad u_\epsilon + u_\epsilon \mapsto -(u_\epsilon + v_\epsilon), \quad u_\epsilon - u_\epsilon \mapsto u_\epsilon - v_\epsilon, \quad \tilde{P}_\epsilon \mapsto -\tilde{P}_\epsilon,$$

and σ_2 is a left-right reflection defined by

$$E_+ \leftrightarrow E_-, \quad u_+ \leftrightarrow v_-, \quad u_- \leftrightarrow v_+, \quad \tilde{P}_+ \leftrightarrow -\tilde{P}_-.$$

These results can be easily expressed in our variables q, R, $[E]$, $\bar{T}_1$, $\bar{E}$ and $[T_1]$ and extended to M and T. Consider the function q; Eqs. (23) yield

$$q = -\mathsf{M}(u_\epsilon + v_\epsilon), \quad \epsilon = \pm 1.$$

It is clear that M is invariant with respect of the whole group; hence, q is invariant with respect to the Λ-transformation for any $\Lambda(E_+, E_-)$, it changes sign by the time reflection σ_1 and is invariant with respect to σ_2. The function R was written in terms of u_ϵ and v_ϵ in Ref. [15] in the form

$$R = \frac{2E_\epsilon}{1 + u_\epsilon v_\epsilon}, \quad \epsilon = \pm 1.$$

Realising that $4 u_\epsilon v_\epsilon = (u_\epsilon + v_\epsilon)^2 - (u_\epsilon - v_\epsilon)^2$, we can see that R is an invariant of all above transformations. The function $[E]$ transforms non-trivially only by σ_2,

$$[E] \mapsto -[E],$$

and $\bar{E}$ is an invariant like R. A comparison of our Eq. (27) with Eq. (43) of Ref. [15] reveals that $T_1^\epsilon = 4\tilde{P}_\epsilon$, so that its Λ-transformation is

$$T_1^\epsilon \mapsto T_1^\epsilon - \epsilon \frac{\partial \Lambda}{\partial E_\epsilon}.$$

Using this formula, one finds easily that

$$\bar{T}_1 \mapsto \bar{T}_1 - \frac{\partial \Lambda}{\partial [E]}, \quad [T_1] \mapsto [T_1] - \frac{\partial \Lambda}{\partial \bar{E}}. \tag{39}$$

The time reflection σ_1 changes the signs of both times:

$$\bar{T}_1 \mapsto -\bar{T}_1, \quad [T_1] \mapsto -[T_1],$$

but the left-right reflection σ_2 acts as follows:

$$\bar{T}_1 \mapsto -\bar{T}_1, \quad [T_1] \mapsto [T_1].$$

From the physical meaning of M and T, it follows that they are invariant with respect to the continuous transformation for any $\Lambda(E_+, E_-)$ and of the left-right reflection σ_2, whereas the time reflection σ_1 must give

$$\mathsf{M} \mapsto \mathsf{M}, \quad \mathsf{T} \mapsto -\mathsf{T}.$$

However, as T is a new cyclic coordinate, the new system has larger continuous symmetry: we can extend Λ to an arbitrary function of the three variables E_+, E_-, and M, and define the action on T as follows:

$$\mathsf{T} \mapsto \mathsf{T} + \frac{\partial \Lambda}{\partial \mathsf{M}}. \tag{40}$$

Let us postpone the proof that this extended Λ-transformation is a symmetry to Sec. VI. It generates three independent constant shifts of the cyclic variables $\bar{T}_1$, $[T_1]$ and T that are different in different shell spacetimes.

We can see from these results that our new variables transform particularly simply. This was, in fact, the idea that helped to find the variables in the first place.

VI. TRANSFORMATIONS OF THE CARTAN FORM

Let us transform the Cartan form Θ in each sector of Γ to the variables q, R, M, T, $[E]$, $\bar{T}_1$ and $[T_1]$ and check that it can then be extended smoothly accross the boundaries of the sectors.

If we substitute for P, T^+, T^- and M, we obtain

$$\Theta = -\bar{E}d[T_1] - [E]d\bar{T}_1 + \mathsf{M}d\mathsf{T} + 2E_+^2 d\left(\ln\left|\frac{u_+}{v_+}\right|\right) - 2E_-^2 d\left(\ln\left|\frac{u_-}{v_-}\right|\right)$$

$$+ \ln\left|\frac{u_+v_-}{u_-v_+}\right| d\left(\frac{R^2}{4}\right) + \ln\left|\frac{u_+}{v_+}\right| d(E_+^2) - \ln\left|\frac{u_-}{v_-}\right| d(E_-^2).$$

The logarithms can be rearranged as follows:

$$\Theta = -\bar{E}d[T_1] - [E]d\bar{T}_1 + \mathsf{M}d\mathsf{T} + d\left(\left(\frac{R^2}{4} + E_+^2\right)\ln\left|\frac{u_+}{v_+}\right| - \left(\frac{R^2}{4} + E_-^2\right)\ln\left|\frac{u_-}{v_-}\right|\right)$$

$$- \left(\frac{R^2}{4} - E_+^2\right) d\left(\ln\left|\frac{u_+}{v_+}\right|\right) + \left(\frac{R^2}{4} - E_-^2\right) d\left(\ln\left|\frac{u_-}{v_-}\right|\right).$$

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The total differential of a function (that diverges badly at the horizons) can be left out. We obtain

$$\Theta = -\bar{E}d[T_1] - [E]d\bar{T}_1 + \mathsf{M}d\mathsf{T} - \left[\left(\frac{R^2}{4} - E^2\right)d\left(\ln\left|\frac{u}{v}\right|\right)\right], \tag{41}$$

where we must substitute for E_+, E_-, $\bar{E}$, u_ϵ and v_ϵ from Eqs. (32), (29) and (23). This form of Θ has been obtained in Ref. [15] (without the matter term); there, however, u_ϵ and v_ϵ were regarded as independent variables. Observe that Eqs. (23), (29) and (32) imply

$$\frac{2E_\epsilon}{R} - 1 = u_\epsilon v_\epsilon \tag{42}$$

so that u_ϵ and v_ϵ are not independent functions for $E_\epsilon = 0$.

Now, we can show that the last term in Θ is regular: we just rewrite it using Eq. (42):

$$\left[\left(\frac{R^2}{4} - E^2\right)d\left(\ln\left|\frac{u}{v}\right|\right)\right] = \left[\frac{R^2}{4}\left(1 + \frac{2E}{R}\right)(vdu - udv)\right]. \tag{43}$$

Thus, our new variables cover all of the constraint hypersurface Γ as promised.

Let us express Θ explicitly by means of the variables q, R, M and $[E]$. Employing Eqs. (23), we have

$$\overline{vdu - udv} = 2\left(-\frac{[E]}{\mathsf{M}^2}dq + \frac{q}{\mathsf{M}^2}d[E]\right),$$

$$[vdu - udv] = 2\left(\frac{1}{R}dq + \frac{q}{R^2}dR - \frac{2q}{R\mathsf{M}}d\mathsf{M}\right).$$

Then, applying the well-known identity $[AB] = \bar{A}[B] + \bar{B}[A]$ that holds for jumps and averages of any two functions A and B, and after using Eq. (29), we arrive at

$$\left[\left(\frac{R^2}{4} - E^2\right)d\left(\ln\left|\frac{u}{v}\right|\right)\right] = \frac{Rq[E]}{\mathsf{M}^2}d[E] - \left(\frac{2Rq}{\mathsf{M}} + \frac{Rq^3}{\mathsf{M}^3} - \frac{Rq[E]^2}{\mathsf{M}^3} - \frac{q\mathsf{M}}{4R}\right)d\mathsf{M}$$

$$+ \left(R + \frac{Rq^2}{2\mathsf{M}^2} - \frac{3R[E]^2}{2\mathsf{M}^2} - \frac{\mathsf{M}^2}{8R}\right)dq + \left(q + \frac{q^3}{2\mathsf{M}^2} - \frac{q[E]^2}{2\mathsf{M}^2} - \frac{q\mathsf{M}^2}{8R^2}\right)dR. \tag{44}$$

This, together with Eq. (41) gives Θ as a function of new variables. It is a complicated one, so we have to transform the coordinates further in order to get a simple expression.

The following question will give some direction to this search for simplicity: is the variable $[T_1]$ suitable for the role of time? More concretely, the time levels in the constraint surface should be *transversal* (this is shown in Ref. [10]). A surface is transversal, if dynamical trajectories intersect it transversally and only once. We can see if the surface $[T_1] = \text{const}$ has this property as follows. The direction of motion on the constraint surface coincides with the direction of degeneration of the presymplectic form $d\Theta$; hence, the pull-back $d\Theta_f$ of $d\Theta$ to any transversal surface $f = \text{const}$ must be non-degenerate. If the surface is $2n$-dimensional, then the $2n$-form $d\Theta_f \wedge \ldots \wedge d\Theta_f$ must be everywhere non-zero. In our case, $n = 3$. Let us calculate $d\Theta_{[T_1]} \wedge d\Theta_{[T_1]} \wedge d\Theta_{[T_1]}$. Eqs. (41) and (44) imply that

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$$d\Theta_{[T_1]} = -d[E] \wedge d\bar{T}_1 + d\mathsf{M} \wedge d\mathsf{T}$$
$$+ A_1 dR \wedge dq + A_2 dR \wedge d[E] + A_3 dR \wedge d\mathsf{M} + A_4 dq \wedge d[E] + A_5 dq \wedge d\mathsf{M} + A_6 d[E] \wedge d\mathsf{M},$$

where A_i, $i = 1, \ldots, 6$ are some functions of the variables R, q, $[E]$, and M. It is clear, therefore, that

$$d\Theta_{[T_1]} \wedge d\Theta_{[T_1]} \wedge d\Theta_{[T_1]} = -6A_1 d[E] \wedge d\bar{T}_1 \wedge d\mathsf{M} \wedge d\mathsf{T} \wedge dR \wedge dq,$$

and the points where the six-form vanishes coincide with those where A_1 vanishes. A_1 can be calculated from Eq. (44) with the result

$$A_1 = \frac{q^2}{\mathsf{M}^2} + \frac{[E]^2}{\mathsf{M}^2} - \frac{\mathsf{M}^2}{4R^2}. \tag{45}$$

Thus, A_1 vanishes for

$$[E]^2 = \frac{\mathsf{M}^4}{4R^2} - q^2.$$

Do these points lie in the physical region that is given by inequality (38)? Let us substitute the value of $[E]^2$ into Eq. (38) and rewrite the result in the form

$$\frac{\mathsf{M}^2}{R^2} \leq 4 \left(1 + \frac{q^2}{\mathsf{M}^2} \right) - \frac{3\mathsf{M}^2}{\mathsf{M}^2 + q^2}.$$

This inequality implies inequality (35); hence, all zero points of A_1 lie in the physical region. We conclude that $[T_1] = $ const is *no* transversal surface.

Let us try to shift $[T_1]$ by a function dependent on the variables R, q, $[E]$ and M:

$$[T_1] = T_2 + X(R, q, [E], \mathsf{M}).$$

Our motivation for choosing such a form is that T_2 is then transformed by the Λ-symmetry (39) in the same way as $[T_1]$ is, and that it is also a cyclic coordinate. Substitution for $[T_1]$ into $d\Theta$ changes the coefficient A_1 at $dR \wedge dq$ in $d\Theta_{T_2}$ by

$$A_1 \rightarrow A_1 + \frac{\partial \bar{E}}{\partial q} \frac{\partial X}{\partial R} - \frac{\partial \bar{E}}{\partial R} \frac{\partial X}{\partial q}.$$

Comparing A_1, Eq. (45), with $\partial \bar{E}/\partial R$ calculated from Eq. (29),

$$\frac{\partial \bar{E}}{\partial R} = \frac{1}{2} + \frac{q^2}{2\mathsf{M}^2} - \frac{[E]^2}{2\mathsf{M}^2} + \frac{\mathsf{M}^2}{8R^2},$$

we observe that

$$A_1 + 2\frac{\partial \bar{E}}{\partial R} = 2 + 2\frac{q^2}{\mathsf{M}^2},$$

which is always positive; thus, one possible shift is

$$[T_1] = T_2 - 2q. \tag{46}$$

If we perform this transformation in Θ, we find out easily that another shift,

$$\bar{T}_1 = T_3 + 2\frac{Rq[E]}{\mathsf{M}^2}, \tag{47}$$

miraculously cancels some pesky cross terms in $d\Theta$. Indeed, shifting the 'times' in the Cartan form, we have:

$$-\bar{E}d[T_1] - [E]d\bar{T}_1 + \mathsf{M}d\mathsf{T} = -\bar{E}dT_2 - [E]dT_3 + \mathsf{M}d\mathsf{T}$$
$$- \left(R + \frac{Rq^2}{\mathsf{M}^2} - \frac{3R[E]^2}{\mathsf{M}^2} - \frac{\mathsf{M}^2}{4R}\right)dq + \frac{2q[E]^2}{\mathsf{M}^2}dR + \frac{2qR[E]}{\mathsf{M}^2}d[E] - \frac{4qR[E]^2}{\mathsf{M}^3}d\mathsf{M}.$$

Using Eqs. (41) and (44), we obtain

$$\Theta = -\bar{E}dT_2 - [E]dT_3 + \mathsf{M}d\mathsf{T} - d\left(\frac{3qR[E]^2}{2\mathsf{M}^2} + \frac{q\mathsf{M}^2}{8R}\right)$$
$$+ \frac{Rq^2}{2\mathsf{M}^2}dq - \left(q + \frac{q^3}{2\mathsf{M}^2}\right)dR + \frac{2R}{\mathsf{M}}\left(q + \frac{q^3}{2\mathsf{M}^2}\right)d\mathsf{M}.$$

The last three terms do not contain other variables than q, R and M and so the terms with $d[E] \wedge dR$, $d[E] \wedge dq$ and $d[E] \wedge d\mathsf{M}$ disappear from $d\Theta$. We have not found any simple physical or geometrical interpretation of the new times T_2 and T_3; observe that the shifts (46) and (47) "mix the sides".

The fact that the last three terms in Θ contain only three variables means that they can be simplified to just one term. For example,

$$\frac{Rq^2}{2\mathsf{M}^2}dq - \left(q + \frac{q^3}{2\mathsf{M}^2}\right)dR + \frac{2R}{\mathsf{M}}\left(q + \frac{q^3}{2\mathsf{M}^2}\right)d\mathsf{M} = d\left(\frac{Rq^3}{6\mathsf{M}^2}\right) - \left(q\mathsf{M}^2 + \frac{2q^3}{3}\right)d\left(\frac{R}{\mathsf{M}^2}\right).$$

This suggests the second step of our transformation:

$$p := q\mathsf{M}^2 + \frac{2q^3}{3}, \tag{48}$$

$$x := \frac{R}{\mathsf{M}^2}. \tag{49}$$

We arrive so at the final shape of the Cartan form

$$\Theta = -\bar{E}dT_2 - [E]dT_3 + \mathsf{M}d\mathsf{T} - pdx, \tag{50}$$

where $\bar{E}$ is given by Eq. (29) in which q and R are expressed in terms of p and x; such expressions can be obtained by solving Eqs. (48) and (49) for q and R.

Consider Eq. (48). If we keep M fixed, p is an increasing function of q in the whole interval $(-\infty, \infty)$, for

$$\frac{\partial p}{\partial q} = \mathsf{M}^2 + 2q^2 > 0.$$

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Hence, the function maps the range $(-\infty, \infty)$ of q onto the range $(-\infty, \infty)$ of p in a bijective way, and it possesses a unique inverse. One can explicitly write down this inverse using second and third roots; this will make $\bar{E}$ a complicated function of p, x, M and $[E]$ (we shall write down this function later, cf. Eq. (63)).

Eq. (50) implies that

$$d\Theta_{T_2} \wedge d\Theta_{T_2} \wedge d\Theta_{T_2} = 6d[E] \wedge dT_3 \wedge d\mathsf{M} \wedge d\mathsf{T} \wedge dp \wedge dx,$$

$$d\Theta_{T_3} \wedge d\Theta_{T_3} \wedge d\Theta_{T_3} = 6\frac{\partial \bar{E}}{\partial [E]} d[E] \wedge dT_2 \wedge d\mathsf{M} \wedge d\mathsf{T} \wedge dp \wedge dx,$$

$$d\Theta_{T} \wedge d\Theta_{T} \wedge d\Theta_{T} = -6\frac{\partial \bar{E}}{\partial \mathsf{M}} d[E] \wedge dT_3 \wedge d\mathsf{M} \wedge dT_2 \wedge dp \wedge dx,$$

and Eq. (29) gives

$$\frac{\partial \bar{E}}{\partial [E]} = -\frac{R}{\mathsf{M}^2}[E],$$

$$\frac{\partial \bar{E}}{\partial \mathsf{M}} = -\frac{\mathsf{M}}{4R}.$$

Thus, the surface $T_3 = \text{const}$ is not transversal at $[E] = 0$, but $T_2 = \text{const}$ and $\mathsf{T} = \text{const}$ are transversal everywhere.

The Cartan form (50) is not yet very simple, because $\bar{E}$ is a complicated function of p, x, $[E]$ and M. What we can still do is to extend the phase space to an eight-dimensional manifold with the cordinates $\bar{E}$, $[T_2]$, $[E]$, $\bar{T}_2$, M, T, p and x, and to express the constraint between these variables as a polynomial. Indeed, applying Eq. (49), we can rewrite Eq. (29) as follows

$$q^2 = \frac{1}{4x^2} + \frac{2\bar{E}}{x} + [E]^2 - \mathsf{M}^2;$$

squaring Eq. (48) and substituting the above expression for q^2 into the result, we obtain

$$p^2 = \frac{1}{9}\left(\frac{1}{4x^2} + \frac{2\bar{E}}{x} + [E]^2 - \mathsf{M}^2\right)\left(\frac{1}{2x^2} + \frac{4\bar{E}}{x} + 2[E]^2 + \mathsf{M}^2\right)^2. \tag{51}$$

This is a constraint that *is* polynomial in the momenta p, $\bar{E}$, $[E]$ and M. The inequalities defining the ranges of the variables $\bar{E}$, $[T_2]$, $[E]$, $\bar{T}_2$, M, T, p and x in the physical region are also simplified:

$$[E]^2 \le 4\bar{E}^2, \quad \mathsf{M} > 0, \quad x > 0. \tag{52}$$

Let us, finally, find the transformation of the variables p, x, M, T, $\bar{E}$, T_2, $[E]$ and T_3 by the symmetry group. The results are as follows. We have for the Λ-transformation,

$$p \mapsto p, \quad x \mapsto x, \quad \mathsf{M} \mapsto \mathsf{M}, \quad \mathsf{T} \mapsto \mathsf{T}, \quad \bar{E} \mapsto \bar{E}, \quad [E] \mapsto [E],$$

$$T_2 \mapsto T_2 - \frac{\partial \Lambda}{\partial \bar{E}}, \quad T_3 \mapsto T_3 - \frac{\partial \Lambda}{\partial [E]}, \quad \mathsf{T} \mapsto \mathsf{T} + \frac{\partial \Lambda}{\partial \mathsf{M}};$$

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for the time reflection σ_1,

$$p \mapsto -p, \quad x \mapsto x, \quad \mathsf{M} \mapsto \mathsf{M}, \quad \mathsf{T} \mapsto -\mathsf{T}, \quad \bar{E} \mapsto \bar{E}, \quad [E] \mapsto [E], \quad T_2 \mapsto -T_2, \quad T_3 \mapsto -T_3;$$

and for the left-right reflection σ_2,

$$p \mapsto p, \quad x \mapsto x, \quad \mathsf{M} \mapsto \mathsf{M}, \quad \mathsf{T} \mapsto \mathsf{T}, \quad \bar{E} \mapsto \bar{E}, \quad [E] \mapsto -[E], \quad T_2 \mapsto T_2, \quad T_3 \mapsto -T_3.$$

Thus, the simple form of the symmetry transformations is preserved by the shifts (46) and (47) as well as by transformations (48) and (49). Eq. (50) implies that

$$\Theta \mapsto \Theta + d\left(\frac{\partial \Lambda}{\partial \bar{E}}\bar{E} + \frac{\partial \Lambda}{\partial [E]}[E] + \frac{\partial \Lambda}{\partial \mathsf{M}}\mathsf{E} - \Lambda\right)$$

by a Λ-transformation, so our extended Λ-transformation *is* a symmetry,

$$\Theta \mapsto -\Theta$$

by the time reflection σ_1,[2] and

$$\Theta \mapsto \Theta$$

by the left-right reflection σ_2. The constraint equation (51) is clearly invariant of the group. Thus, the action of the symmetry group is well-defined in the whole phase space and it preserves the physical region.

VII. THE CONSTRAINT SURFACE

In this section, we are going to study the properties of the constraint (51). We are interested in the topology of the constraint surface; we shall investigate the salient question of whether or not any spurious solutions have been unintentionally included during the process of transforming the constraint to the polynomial form; and we shall look for solutions of the constraint (51) with respect to some momenta.

Our new variational principle reads

$$S = \int dt(-\bar{E}\dot{T}_2 - [E]\dot{T}_3 + \mathsf{M}\dot{\mathsf{T}} - p\dot{x} - \nu_1 \mathcal{C}_1), \tag{53}$$

where

$$\mathcal{C}_1 = 9p^2 - \left(\frac{1}{4x^2} + \frac{2\bar{E}}{x} + [E]^2 - \mathsf{M}^2\right)\left(\frac{1}{2x^2} + \frac{4\bar{E}}{x} + 2[E]^2 + \mathsf{M}^2\right)^2. \tag{54}$$

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[2]The sign change of Θ by σ_1 is necessary, because time reflections are anti-symplectic transformations.

The constraint surface is trivial in the direction of the three times T_2, T_3 and T, so we can limit ourselves to its projection to the five-dimensional space spanned by $\bar{E}$, $[E]$, M, p and x. Let us observe that $\mathcal{C}_1$ depends on $\bar{E}$, $[E]$ and x through a function that we shall call B,

$$B := \frac{1}{4x^2} + \frac{2\bar{E}}{x} + [E]^2, \tag{55}$$

and that the gradient of B is non-zero. It is, therefore, advantageous, to write $\mathcal{C}_1$ as a composed function,

$$\mathcal{C}_1 = 9p^2 - (B - \mathsf{M}^2)(2B + \mathsf{M}^2)^2.$$

Let us first look for the singular points of the constraint surface (that is, where the gradient of $\mathcal{C}_1$ vanishes). We have

$$\frac{\partial \mathcal{C}_1}{\partial p} = 18p, \tag{56}$$

$$\frac{\partial \mathcal{C}_1}{\partial \mathsf{M}} = 6\mathsf{M}^3(2B + \mathsf{M}^2), \tag{57}$$

$$\frac{\partial \mathcal{C}_1}{\partial B} = -3(2B + \mathsf{M}^2)(2B - \mathsf{M}^2). \tag{58}$$

As all three derivatives must vanish at a singular point, all such points are determined by the equations

$$p = 0, \quad 2B + \mathsf{M}^2 = 0. \tag{59}$$

All points satisfying Eqs. (59) are solutions of Eq. (51); however, the physical ranges (52) of the variables $\bar{E}$, $[E]$ and x do not allow B to be negative. We conclude that the constraint surface is regular (smooth) in the physical region (52).

The topology of the constraint surface can be found in the shortest way, if we consider $\mathcal{C}_1$ as a function $\mathcal{C}_1(B)$ of B keeping all other variables constant. Studying Eqs. (54) and (58) we can easily see that $\mathcal{C}_1(B)$ decreases in the interval $(-\infty, -\mathsf{M}^2/2)$ from ∞ to $9p^2$, it increases in the interval $(-\mathsf{M}^2/2), \mathsf{M}^2/2)$, from $9p^2$ to $9p^2 + 2\mathsf{M}^6$ and it decreases again in the interval $(\mathsf{M}^2/2, \infty)$ from $9p^2 + 2\mathsf{M}^6$ to $-\infty$. It follows that there is only one solution for $p \neq 0$ (because $\mathcal{C}_1(B) = 9p^2$ for $B = \mathsf{M}^2$). As $\mathcal{C}_1(\mathsf{M}^2) = 9p^2 > 0$, the solution must satisfy the inequality $B > \mathsf{M}^2$. This solution depends smoothly on the values of all other variables. For $p = 0$, there are, however, two solutions: one with $B = \mathsf{M}^2$, which is a continuous extension of the previous $p \neq 0$ case, and a new one with $B = -\mathsf{M}^2/2$, which appears when the local minimum at $-\mathsf{M}^2/2$ touches the B-axis. This shows that the constraint surface consists of two components: one is a seven-dimensional smooth manifold, let us denote it by Γ_7, that lies in the region $B \geq \mathsf{M}^2$, and one is a six-dimensional smooth manifold, let us denote it by Γ_6, that is defined by Eqs. (59).

Next, let us study the solvability of the constraint equation (51) with respect to the conserved momenta. The unique solution of the constraint (51) with respect to B at Γ_7 is given by

$$B = f(p, \mathsf{M}), \tag{60}$$

where

$$f(p, \mathsf{M}) := \frac{1}{2}\sqrt[3]{9p^2 + \mathsf{M}^6 + 3p\sqrt{9p^2 + 2\mathsf{M}^6}} + \frac{1}{2}\sqrt[3]{9p^2 + \mathsf{M}^6 - 3p\sqrt{9p^2 + 2\mathsf{M}^6}}. \tag{61}$$

The equation (51) has, in fact, three independent solutions, but only one of them is real. For $p = 0$, B reaches its minimum M^2 on Γ_7; thus, at Γ_7,

$$B \geq \mathsf{M}^2. \tag{62}$$

Given B, one can solve uniquely for $\bar{E}$ and there is also a unique solution for $[E]^2$. The solution for $\bar{E}$ is

$$\bar{E} = -\frac{1}{8x} - \frac{x[E]^2}{2} + \frac{x}{2}f(p, \mathsf{M}). \tag{63}$$

Similarly, we can find the solution of the constraint (51) with respect to M. Consider $\mathcal{C}_1$ as a function $\mathcal{C}_1(\mathsf{M})$ of M keeping all other variables fixed. $\mathcal{C}_1(\mathsf{M})$ is symmetric with respect to the reflection on the $\mathcal{C}_1$-axis. We have to distinguish two cases: $B < 0$, $B = 0$: $\mathcal{C}_1(\mathsf{M})$ decreases in the interval $(-\infty, 2B)$ from ∞ to $9p^2$, it increases in the interval $(2B, 0)$ from $9p^2$ to the local maximum $9p^2 - 4B^3$ at $\mathsf{M} = 0$, it decreases again in the interval $(0, -2B)$ from $9p^2 - 4B^3$ to $9p^2$, and, finally, it increases in $(-2B, \infty)$ from $9p^2$ to ∞. A solution exists only for $p = 0$, and there are two solutions, then, $\mathsf{M} = \pm\sqrt{-2B}$. This is at the surface Γ_6. $B > 0$: then, there is only one minimum, at $\mathsf{M} = 0$, with the value $9p^2 - 4B^3$; there are no other extrema. Consider Eq. (51) as an equation for M^{-2}. There are three real solutions for M^{-2}, but all of them are negative unless

$$4B^3 - 9p^2 \geq 0, \tag{64}$$

and then there is only one non-negative solution. A reasonable solution is, therefore, unique, it depends continuously of the other variables and it lies at Γ_7. We can write it in the form

$$\mathsf{M} = \pm\sqrt[4]{\frac{4B^3 - 9p^2}{4B}}\left(\cos\frac{1}{3}\arctan\frac{3p}{\sqrt{4B^3 - 9p^2}}\right)^{-1/2}.$$

This solution can, of course, be also expressed by means of square and third roots, but only if one employs complex numbers. We can see that squaring of M introduced non-physical solutions with negative M; we have to choose the positive branch. Observe that all points at the surface Γ_7 must satisfy the inequalities (62) and (64).

The equations of motion that follow from the action (53) can be written as follows

$$\dot{\bar{E}} = [\dot{E}] = \dot{\mathsf{M}} = 0, \tag{65}$$

$$\dot{T}_2 = -\nu_1\frac{\partial\mathcal{C}_1}{\partial B}\frac{\partial B}{\partial\bar{E}}, \tag{66}$$

$$\dot{T}_3 = -\nu_1\frac{\partial\mathcal{C}_1}{\partial B}\frac{\partial B}{\partial[E]}, \tag{67}$$

$$\dot{\mathsf{T}} = \nu_1\frac{\partial\mathcal{C}_1}{\partial\mathsf{M}}, \tag{68}$$

$$\dot{p} = \nu_1\frac{\partial\mathcal{C}_1}{\partial B}\frac{\partial B}{\partial x}, \tag{69}$$

$$\dot{x} = -\nu_1\frac{\partial\mathcal{C}_1}{\partial p}. \tag{70}$$

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It is easy to derive the radial equation from them. On Γ_6, we have only a static solutions:

$$\dot{x} = \dot{T} = \dot{p} = \dot{T}_2 = \dot{T}_3 = 0.$$

On Γ_7, $\dot{T} > 0$ for $\nu_1 > 0$ and Eqs. (57), (68) and (70) yield

$$\frac{dx}{d\mathsf{T}} = -\frac{3p}{\mathsf{M}^3(2B + \mathsf{M}^2)}.$$

Using the constraint (51) to exclude p, we obtain the radial equation on Γ_7:

$$\frac{dx}{d\mathsf{T}} = \pm\frac{1}{\mathsf{M}^2}\sqrt{\frac{B}{\mathsf{M}^2} - 1}. \tag{71}$$

Let us compare it with Eq. (13). Eqs. (65) and (49) imply that

$$\frac{dx}{d\mathsf{T}} = \frac{1}{\mathsf{M}^2}\frac{dR}{d\mathsf{T}},$$

whereas Eqs. (55), (49) and (14) give

$$V = 1 - \frac{B}{\mathsf{M}^2}.$$

Hence, the radial equations (71) and (13) are equivalent, and the dynamics of the new action on Γ_7 coincides with that of the old action. Γ_6 consists of the unintentionally added unphysical solutions, at least on the phase space defined by Eqs. (18), but nothing is added in the physical region.

Finally, we shall study the question of the monotonicity of the time functions T_2, T_3 and T along the dynamical trajectories at Γ_7. Eqs. (57) and (58) show that the right-hand sides of Eqs. (66) and (68) cannot vanish, and we have for $\nu_1 > 0$:

$$\dot{T}_2 > 0, \quad \dot{\mathsf{T}} > 0.$$

Hence, T_2 and T are good times and, as we have seen above, the constraint can be solved uniquelly for the corresponding momenta, $\bar{E}$ and M. As for T_3, Eq. (67) implies that $\dot{T}_3 > 0$ for $[E] > 0$, $\dot{T}_3 = 0$ for $[E] = 0$ and $\dot{T}_3 < 0$ for $[E] < 0$. The solvability with respect to $[E]$ is also only partial: there is a unique positive and a unique negative solution. Thus T_3 is a good time only in some special cases, as the next section will show.

VIII. SOME INTERESTING SPECIAL CASES

In this section, we shall remove some degrees of freedom and describe the resulting simpler models in terms of the new variables.

A. Dust degrees of freedom removed

Here, we remove the variables M and T and return to the system considered in Ref. [14]. It has two degrees of freedom: the black hole mass E_- and the position x of the shell. To this aim, we first choose a particular value of M and demote it so to a mere parameter. In this way, a submanifold Γ_{7M} of Γ_7 emerges. Second, we take a quotient of Γ_{7M} by the T-curves (see Ref. [15] for more detail of this *exclusion of a cyclic variable*). The system that results has the action

$$S_M = \int dt (-\bar{E}\dot{T}_2 - [E]\dot{T}_3 - p\dot{x} - \nu_1 \mathcal{C}_1),$$

where $\mathcal{C}_1$ is given by Eq. (54) as before, only M is a parameter now. The equations of motion comprise Eq. (65) without $\dot{\mathsf{M}} = 0$, as well as Eqs. (66), (67), (69) and (70).

The properties of this system are analogous to that of the original one. In particular, the constraint is regular everywhere on Γ_{7M} because the derivative (56) and (58) cannot both simultaneously vanish, T_2 is a good time variable and the constraint equation is uniquely solvable for $\bar{E}$.

B. Black hole degree of freedom removed

Here, we demote the variable E_- to a parameter and remove the corresponding cyclic variable $T_2/2 - T_3$; the remaining two degrees of freedom are the internal energy M and the position x of the shell. The system obtained in this way will describe the dynamics of the dust shell in the field of a fixed black hole if $E_- > 0$ or in the Minkowski space if $E_- = 0$. One returns so to the system considered in Ref. [13]; the procedure, in different coordinates, has been performed in Ref. [15].

To begin with, we transform variables as follows:

$$\bar{E} = \frac{E_+ + E_-}{2}, \quad T_2^+ = T_2/2 + T_3,$$

$$[E] = E_+ - E_-, \quad T_2^- = T_2/2 - T_3.$$

Then, we choose a particular value for E_-; this defines the black hole mass, and formally, a six-dimensional submanifold Γ_{7E} of Γ_7. Finally, we take the quotient Γ_{7E}/T_- of Γ_{7E} by the T_--curves. The result is the action

$$S_E = \int dt (-E_+ \dot{T}_2^+ + \mathsf{M}\dot{\mathsf{T}} - p\dot{x} - \nu_1 \mathcal{C}_2),$$

where

$$\mathcal{C}_2 = 9p^2 - (B_2 - \mathsf{M}^2)(2B_2 + \mathsf{M}^2),$$

and

$$B_2 = \frac{1}{4x^2} + \frac{E_+ + E_-}{x} + (E_+ - E_-)^2.$$

The variable T is a good time and $C_2 = 0$ is of course solvable for M exactly as in the general case. The variable T_2^+ is, for large values of the parameter E_-, not a good time, because it is a combination $T_2/2 + T_3$ of T_2 that increases along all dynamical trajectories and T_3 that increases along those with $E_+ > E_-$ and decreases along those with $E_+ < E_-$. One can show that $\dot{T}_2^+$ is positive for $E_+ - E_- > 0$ and negative for $E_+ - E_- < -\mathsf{M}$ and that there is no surface for $E_- > \mathsf{M}$ that would be transversal to both the dynamical trajectories and the orbits of the continuous subgroup. [3] The derivative of $\mathcal{C}_2$ with respect to E_+ changes sign somewhere if E_- is sufficiently large and so there are more that one, or no solution of the constraint with respect to E_+.

There is one prominent exception: suppose that $E_- = 0$. Then, $E_+ - E_-$ is positive if E_+ is. Hence, in the physical region (where $E_+ > 0$), T_2^+ *is* a good time and the constraint has a *unique* solution for E_+ there, namely

$$E_+ = -\frac{1}{2x} + \sqrt{f(p, \mathsf{M})},$$

where f is defined by Eq. (61).

One can also remove all four variables T, M, E_- and T_2^- so that only one degree of freedom, the position x of the shell remains, but it is not difficult to work out the properties of the system using the results obtained in this section, and we left it as an easy exercise to the reader.

The special case $E_- = 0$ is also interesting, because it is a part of the boundary of the physical region given by inequalities (52) on Γ_7. The variational principle (53) does not break down at the boundary, that is, it defines a regular dynamics there. From this, it must clearly follow that the action defines a regular dynamics even at those points of Γ_7 that do not satisfy the conditions (52). We have studied the meaning of this dynamics and of these points in Sec. II.

IX. ACKNOWLEDGMENTS

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[3]Still, one can transform T_2^+ so that the resulting time levels are transversal to the dynamical trajectories, but a mere shift is not sufficient; one must screw the level to a helix form with the helix axis in the E_+-direction; this may be done by a singular "shift" proportional to $(E_+ - E_-)^{-1}$. We shall not go into detail.

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